MATH3030 Group theory studying guide

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The study of group theory involves two paths that often intersect. Along the first path, one studies general properties of arbitrary (finite) groups, while along the second, one studies some particular groups that are interesting or fundamental. The two paths often intertwine with each other, and one often needs knowledge from one path to move on along the other. Let's summarize the properties that one will encounter in the theory part, and also as many examples as you can think of.

1 Properties of groups

1. Order of group
2. Orders of elements
3. Subgroups
4. Abelian, cyclic
5. Homomorphisms and isomorphisms
6. Generators
7. **(Generators and) Relations**
8. **Normals subgroups, quotient groups=Homomorphic images**
9. **Center and commutator**
10. **Derived Series (Subnormal series, solvable groups)**
11. **Group actions, conjugate classes**
12. **Sylow p-groups**
13. Representations, character table

The bold items will be what you will learn in MATH3030. Items 1-6 was in MATH2070, and the last item will be studied in MATH4080. The ordering of the items is chronological, that is, by when it first appears in courses.

Items 1,2,4,6,7,9,10,12 are most intrinsic. Items 3,5,8 describes homomorphisms, subgroups, and images, and they describe the relations between two groups. Items 11, 13 studies the actions of groups on a set and on a linear space respectively, and this approach to studying groups is indirect while very powerful.

Given a group G, the first question is to ask how complex it is. One might first look at its order (1), and check how abelian it is (4). If it is abelian, then it is very well controlled (see 2.?). If not, we calculate its center $Z(G)$ and commutator $G' = (G, G)$ (9). If the center is large with respect to G, or if the commutator is small, then G is not too far from an abelian group. One perform G' repeatedly, if one gets 1 at some step, then group is solvable (10). Sylow p-subgroups (12) form an important class of subgroups, and it provides some information of groups of prime power order. With these, one can handle groups of medium size and complexity, such as some groups of order $p^k q$, pqr, where p, q, r are distinct primes. Group actions (11) and representation theory (13) are more power tools that allow one study groups of even larger sizes. One celebrated theorem of Burnside states that groups of order $p^a q^b$ are solvable, and its proof uses representation theory and is slightly beyond MATH4080.

As the group sizes grow, it is harder and harder to give a precise characterization, and more complex tools are needed to study the group in a relatively indirect way. For easy groups like $C_n \simeq \mathbb{Z}/n\mathbb{Z}, D_n$, it is suitable to ask about the orders of its elements, and to write down all there subgroups, but for larger groups, it is suitable to first do for example the center-commutator-derived series analysis.

To conclude, we will ask the following properties for a finite group: order, $Z(G)$, G', cyclic/abelian/solvable, Subgroups ($p-$subgroups), generator and relations.

2 Examples of groups

The examples of groups come from different origins

2.1 Geometric symmetries

Examples are: $C_n, D_n, S_n, A_n, \ldots.$

The cyclic group C_n of order n is the rotation group of a regular n-gon. The dihedral group D_n of order $2n$ is the rotation and reflection group of a regular n-gon. S_n and A_n also have geometric characterizations.

2.2 Automorphism group of other structures

Examples are: S_X for a set X, $\mathrm{Aut}(G)$ for a group G, $\mathrm{Aut}(R)$ for a ring R, $\mathrm{Aut}(E/F)$ for a field extension E/F, $\mathrm{GL}_F(V)$ for a vector space V over a field F, ...

2.3 $R^\times$ for a ring R

Examples are: $(\mathbb{Z}/n\mathbb{Z})^\times, M_n(R)^\times = \mathrm{GL}_n(R), \mathbb{F}_{p^n}^\times$.

The structure of $(\mathbb{Z}/n\mathbb{Z})^\times$ is an important topic in number theory (MATH3080). That $\mathbb{F}_{p^n}^\times$ is cyclic is the starting point of studying the structure of finite fields (MATH2070 and MATH3040).

2.4 Subgroups of $\mathrm{GL}_n(F)$

$D_n(F)$ for invertible diagonal matrices. $B_n(F)$ for invertible upper triangular matrices. $\mathrm{SL}_n(F)$ for matrices with determinant 1. $\mathrm{O}_n(F)$ for matrices with $A^T A = I$.

One may also replace the field F by a ring R such as $\mathbb{Z}/p^k\mathbb{Z}$ to get more examples.

2.5 Constructions

These are groups constructed from other groups via the following constructions:

Direct products such as $\prod_i C_{d_i}$. In fact, all finite abelian groups come in this way.

Semi-direct products such as $\mathbb{Z}_q \rtimes \mathbb{Z}_p$ for $p \mid q-1$.

Subgroups: In fact, any finite group is isomorphic to a subgroup of S_n. (Cayley's theorem)

Quotients: $\mathrm{PGL}_n(F) := \mathrm{GL}_n(F)/F^\times$, ...

Generator and relations: Free groups, free abelian groups $\mathbb{Z}^I$, ...

3 Table

G	$\|G\|$	$Z(G)$	$G' = (G,G)$	cyc/ab/solv	Subgroups (normal or $p-$subgroups)	Gen. and rel.	Conjugate Classes
C_n	n	C_n	1	cyclic	$\langle g^d \rangle$ for each $d\|n$	$\langle g \mid g^n = 1 \rangle$	Each element g forms a class $[g]$
D_n, $n \geq 3$ odd	$2n$						
D_n, $n \geq 4$ even	$2n$						
S_3	6						
S_4	24						
$S_n, n \geq 5$	$n!$						
$A_n, n \geq 5$	$n!/2$						
$\mathrm{GL}_n(\mathbb{F}_p)$	$\prod_{i=0}^{n-1} (p^n - p^i)$						